Formalization of Laplace Transform using the Multivariable Calculus Theory of HOL-Light

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Outline

- Introduction
- Motivation
- Formalization Details
- Case Study
  - Linear Transfer Converter (LTC) circuit
- Conclusions

Laplace Transform

- Integral transform method
  - *Pierre Simon Laplace 1749-1827*

- Mathematically represented by the following improper integral

$$F(s) = \int_0^\infty f(t)e^{-st}dt, \ \ s \in \mathbb{C}$$

- A linear operator
  - Input: Time varying function, i.e., a function $f(t)$ with a real argument t ($t \geq 0$)
  - Output: $F(s)$ with complex argument s

Laplace Transform - Key Benefits and Utilizations

- Solve linear Ordinary Differential Equations (ODEs) using simple algebraic techniques

- Obtain concise and useful input/output relationships (Transfer Functions) for systems
  - Widely used in Control System and Analog Circuit Design

Laplace Transform - Example

$$\frac{d^2y(t)}{dt^2} + 4y(t) = x(t)$$

$$\mathcal{L}\Big(\frac{d^2y}{dt^2} + 4y(t)\Big)s = \mathcal{L}x(t)s$$ Taking Laplace Transform on Both sides

$$s^2(\mathcal{L}y(t)s) + 4(\mathcal{L}y(t)s) = (\mathcal{L}x(t)s)$$ Using the Laplace of a differential and the Linearity of Laplace Properties

$$\frac{(\mathcal{L}y(t)s)}{(\mathcal{L}x(t)s)} = \frac{1}{s^2+4}$$ Transfer Function

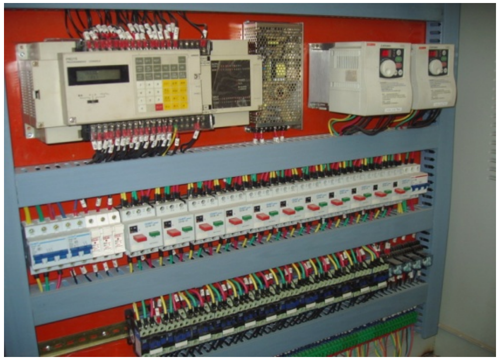
$$\text{let } x(t) = sin(2t), \text{ then } \mathcal{L}x(t)(s) = \frac{2}{s^2+4}$$ Laplace of sine

$$\mathcal{L}y(t)s = \frac{2}{(s^2+4)^2} \xrightarrow{\text{Inverse Laplace}} y(t) = \frac{1}{8}sin(2t) - \frac{t}{4}cos(2t)$$

Solution in time domain

Real-World Applications for Laplace Transforms

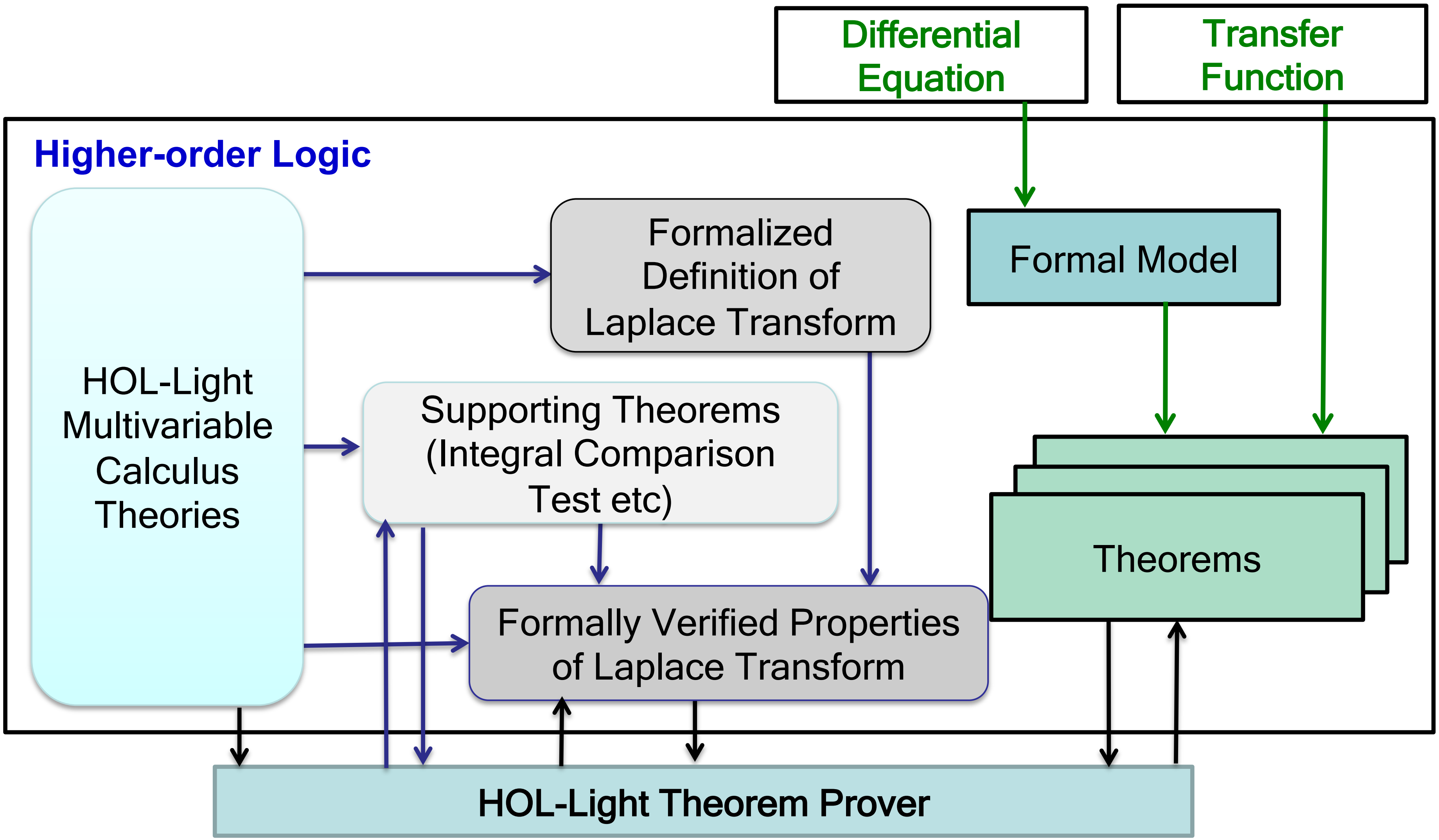
- Integral part of analyzing many physical systems
  - Aerodynamic systems
  - Circuit Analysis
  - Control systems
  - Mechanical networks
  - Analogue filters



Laplace Transform based Analysis

Criteria	Paper-and-Pencil Proof	Simulation/ Symbolic Methods	Automated Formal Methods (MC, ATP)	Computer Algebra Systems	Higher-order-logic Proof Assistants
Expressiveness	✓	✓	✗	✓	✓
Accuracy	✓ ?	✗	✓	✗	✓
Automation	✗	✓	✓	✓	✗

Proposed Approach for Verifying Transfer Functions



Formal Definition of Laplace Transform

❑ Mathematical definition

$$F(s) = \int_0^\infty f(t)e^{-st}dt, \; s \in \mathbb{C} \Longrightarrow F(s) = \lim_{b \to \infty} \int_0^b f(t)e^{-st}dt$$

Given f is piecewise smooth and is of exponential order i.e. there exist constants $\alpha \in \mathbb{R}$ and $0 < M$ such that $|f(t)| \leq Me^{\alpha t}$ for all $t \geq 0$

Definition : Laplace Transform

```
⊢ ∀ s f. laplace f s =
  lim at_posinfinity (λb. integral (interval [lift(&0),lift(b)])
    (λt. cexp (-(s * Cx(drop t))) * f t))
```

Definition : Conditions for Laplace Existence

```
⊢ ∀ s f. laplace_exists f s ⇔
(∀ b. f piecewise_differentiable_on interval [lift (&0),lift b])
  ∧ (∃ M a. Re s > drop a ∧ exp_order f M a)
```

Formalized Laplace Transform Properties

Property	Mathematical Form
Limit Existence	$\exists l. \int_0^{\infty} f(t) e^{-st} dt = l$
Linearity	$\mathcal{L}\left(\alpha f(t) + \beta g(t)\right) s = \alpha(\mathcal{L} f(t) s) + \beta(\mathcal{L} g(t) s)$
Frequency Shifting	$\mathcal{L}\left(e^{bt} f(t)\right) s = \mathcal{L}\ f(t)(s - b)$

Property	Formalized Form
Limit Existence	⊢ ∀ f s. laplace_exists f s ⇒ (∃l. ((λb. integral (interval [lift (&0),lift b]) (λt. cexp (-(s * Cx (drop t))) * f t)) → l) at_posinfinity)
Linearity	⊢ ∀ f g s a b. laplace_exists f s ∧ laplace_exists g s ⇒ laplace (λx. a * f x + b * g x) s = a * laplace f s + b * laplace g s
Frequency Shifting	⊢ ∀ f s b. laplace_exists f s ⇒ laplace (λt. cexp (b * Cx (drop t)) * f t) s = laplace f (s - b)

Formalized Laplace Transform Properties

Property	Mathematical Form
Integration	$\mathcal{L}(\int_0^t f(\tau)d\tau)s = \frac{1}{s}(\mathcal{L}f(t)s)$
n-order Differentiation	$\mathcal{L}(\frac{d^n f(t)}{dx^n})s = s^n(\mathcal{L}f(t)s) - \sum_{k=1}^{n} s^{k-1}\frac{d^{n-k}f(0)}{dx^{n-k}}$

Property	Formalized Form
Integration	`⊢ ∀ f s. (&0 < Re s) ∧ laplace_exists f s ∧ laplace_exists (λx. integral (interval [lift (&0),x]) f) s ∧ (∀x. f continuous_on interval [lift (&0),x]) ⇒ laplace (λx. integral (interval [lift (&0),x]) f) s = inv(s) * laplace f s`
Higher Order Differentiation	`⊢ ∀ f s n. laplace_exists_higher_derivative n f s ∧ (∀x. higher_derivative_differentiable n f x) ⇒ laplace (λx. higher_order_derivative n f x) s = s pow n * laplace f s - vsum (1..n) (λx. s pow (x-1) * higher_order_derivative (n-x) f (lift (&0)))`

- 5000 lines of HOL-Light code and approximately 800 man-hours

Case Study: Linear Transfer Converter (LTC) circuit

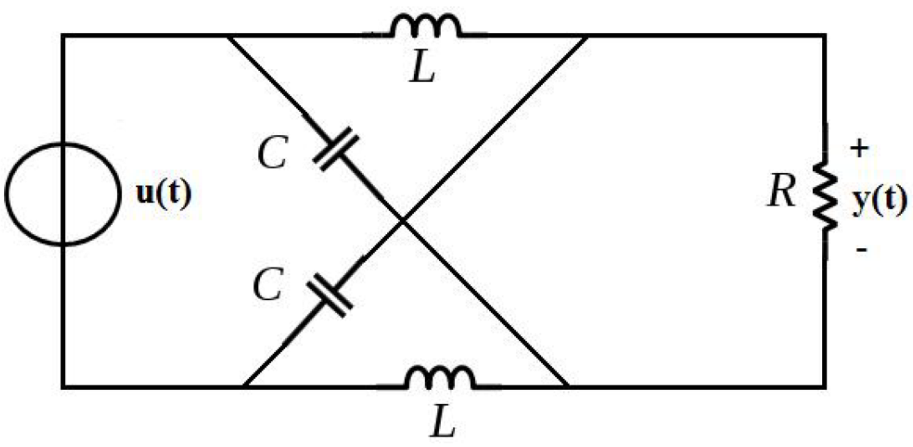
- Converts the voltage and current levels in power electronics systems
- Functional correctness of power systems depends on design and stability of LTC

Differential Equation:

$$\frac{d^2y}{dt^2} - \frac{2}{RC}\frac{dy}{dt} + \frac{1}{LC}y = \frac{d^2u}{dt^2} - \frac{1}{LC}u$$

Transfer Function:

$$\frac{Y(s)}{U(s)} = \frac{s^2 - \frac{1}{LC}}{s^2 - \frac{2s}{RC} + \frac{1}{LC}}$$



Linear Transfer Converter (LTC) circuit

Differential Equation: $$\frac{d^2y}{dt^2} - \frac{2}{RC}\frac{dy}{dt} + \frac{1}{LC}y = \frac{d^2u}{dt^2} - \frac{1}{LC}u$$

Definition : Differential Equation of LTC

```
⊢ ∀ y u x L C R. diff_eq_LTC y u x L C R ⇔
diff_eq 2 [ Cx (&1 / L * C); --Cx (&2 / R * C); Cx (&1)] y x =
diff_eq 2 [ --Cx (&1 / L * C ); Cx (&0); Cx (&1)] u x
```

Definition : Differential Equation

```
⊢ ∀ n L f x. diff_eq n L f x ⇔
  vsum (0..n) (λt. EL t L x * higher_order_derivative t f x)
```

Linear Transfer Converter (LTC)

Theorem : Transfer Function of LTC

```
⊢ ∀ y u s R L C. (&0 < R) ∧ (&0 < L) ∧ (&0 < C) ∧
  (zero_initial_conditions 1 u) ∧ ( zero_initial_conditions 1 y) ∧
   (∀x. higher_derivative_differentiable 2 y x) ∧
   (∀x. higher_derivative_differentiable 2 u x) ∧
   (higher_derivative_laplace_exists 2 y s) ∧
   (higher_derivative_laplace_exists 2 u s) ∧
  ~((Cx(&1/(L*C)) - Cx(&2/(R*C))*s) + s pow 2 = Cx(&0) )∧
  ~(laplace u s = Cx(&0)) ∧ (∀t. diff_eq_LTC y u t L C R) ⇒
    (laplace y s / laplace u s =
     (s pow 2 - Cx(&1/(L*C))) / ((Cx(&1/(L*C)) -
       Cx(&2/(R*C))*s) + s pow 2))
```

- 650 lines of HOL-Light code and the proof process took just a couple of hours

Conclusions

- Formalization of Laplace transform theory using higher-order logic
  - Multivariable Calculus Theory of HOL-Light
- Advantages
  - Accurate Results
  - Reduction in user-effort while formally analyzing Physical Systems that involve Differential Equations
- Case Study: Transfer function verification of LTC circuit

Future Directions

- Application of Laplace transform theory in Analog and Mixed Signal circuits and controls engineering

- Formalization of Inverse Laplace transform

- Formalization of Fourier transform

Thanks!

- For More Information
  - Visit our website
    - http://save.seecs.nust.edu.pk
  - Contact
    - osman.hasan@seecs.nust.edu.pk

Additional slides

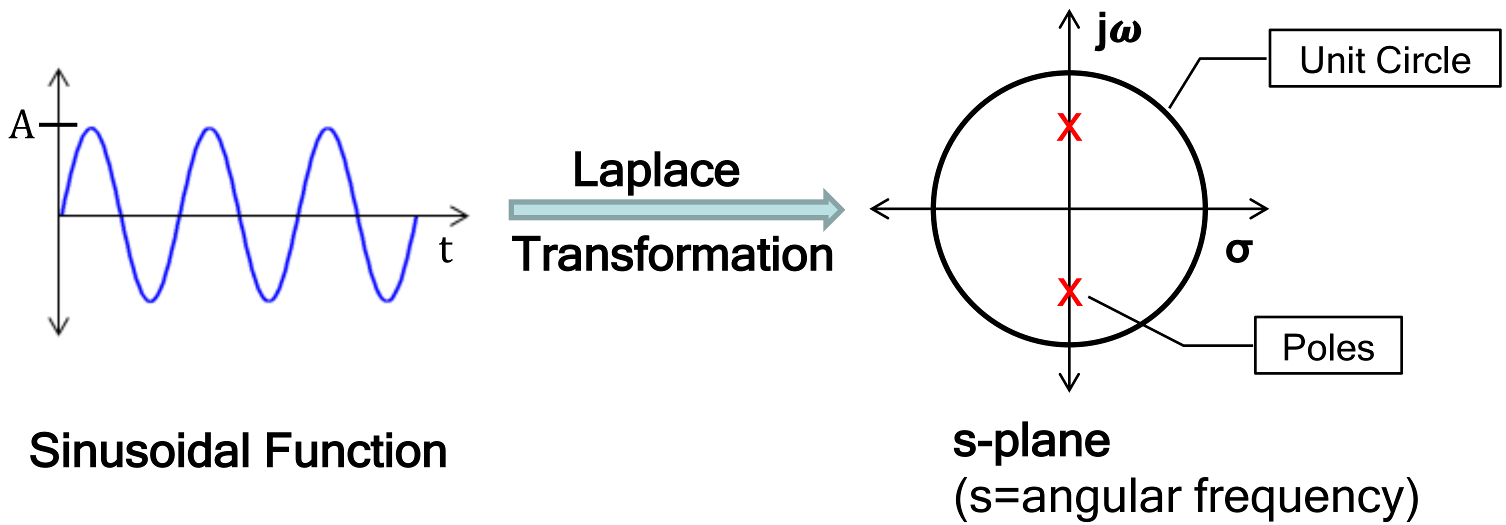
Formalized Laplace Transform

Definition 3: Exponential Order Function

$\vdash \forall$ f M a. exp_order f M a $\Leftrightarrow$ &0 $<$ M $\wedge$
($\forall$ t. &0 $\leq$ t $\Rightarrow$ norm (f (lift t)) $\leq$ M * exp (drop a * t))

Laplace Transform

- Provides compact representation of the overall behavior of the given time varying function



- s-plane representation depicts frequency and phase of sinusoidal signal

HOL-Light

- Multivariable calculus theories
  - Integral theory
  - Differential theory
  - Transcendental theory
  - Topological theory
  - Complex analysis theory
- Real number theory
- Natural number theory

Limit Existence of Laplace Transform

Proof Steps

Split the complex integrand into real and imaginary parts

∃l.((λb. integral (interval [lift (&0),lift b])
(λt. Cx (Re (cexp (-(s * Cx (drop t))) * f t))) +
ii * integral (interval [lift (&0),lift b])
(λt. Cx (Im (cexp (-(s * Cx (drop t))) * f t)))) → l)
at_posinfinity

Convert both complex integrals to their corresponding real integral and split the complex limit to both integrals

laplace_exists f s ⇒
∃k. ((λb. real_integral (real_interval [&0,b])
(λx. abs (Re (cexp (-s * Cx (x)) * f(lift x))))) → k)
at_posinfinity

Using fo… compari… converg…

Lemma 3: Comparison Test for Improper Integrals

∧ f x ≤ g x) ∧
) ∧
) ∧
g)⇒ k)
,b]) f) ⇒ k)

Lemma 2: Limit of a Complex-Valued Function …mplex Integral

⊢ ∀ f L1 L2.
((λt. Re (f t)) ⇒ L1) at_posinfinity ∧
((λt. Im (f t)) ⇒ L2) at_posinfinity ⇒
(f → complex (L1,L2)) at_posinfinity

…val [&0,t]) ⇒

In our…